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# Second-order Nonlinear PDE-based Image Restoration Scheme

**Tudor Barbu**
Institute of Computer Science of the Romanian Academy
T. Codrescu 2, Iaşi
tudor.barbu@iit.academiaromana-is.ro

***Abstract***– An effective nonlinear second-order PDE image denoising and restoration model is described in this paper. The proposed anisotropic diffusion-based filtering scheme is based on some novel versions of the edge-stopping function and the conductance parameter. A consistent numerical approximation scheme is constructed for this continuous model. Our PDE-based smoothing technique provides an efficient noise removal while preserving the edges and other image features. It outperforms both the conventional filters and also many PDE-based denoising approaches, as it results from our successfully experiments and method comparison.

***Keywords*:** PDE-based image restoration, anisotropic diffusion, diffusivity function, conductance parameter, edge preservation, numerical approximation scheme.

## 1. Introduction

During the past 25 years, the differential models have been widely used in several traditionally engineering areas, such as image processing and analysis, and computer vision. The partial differential equations (PDEs) have been successfully utilized for solving many image processing and computer vision problems. In recent years, many image processing and analysis techniques using variational and PDE-based algorithms have been developed, because of the modeling flexibility and some advantages of the numerical implementation of the PDEs (Guichard et al., 2001).

Feature-preserving image restoration represents still a focus in the image processing domain, remaining a serious challenge for researchers. The conventional 2D denoising techniques, like average, median and 2D Gaussian filter may reduce the noise, but also have the disadvantage of blurring the edges and have no localization property (Jain, 1989). For this reason, a lot of edge-preserving approaches based on PDEs have been introduced in the last decades (Barbu, 2013, Guichard et al., 2001).

Many nonlinear second-order diffusion-based noise removal schemes have been proposed since the influential framework of P. Perona and J. Malik (1987), representing an anisotropic diffusion model for image denoising. Since it is common to derive a PDE-based model from a variational problem, numerous variational restoration techniques have been also constructed in the last three decades (Chan et al., 2003).

The most influential variational image smoothing technique is that developed by Rudin, Osher and Fetami (1992). These variational and second-order PDE models overcome the blurring effect but often generate the staircase effect (Buades et al., 2006).

In this paper we propose a novel image restoration approach using a nonlinear second-order anisotropic diffusion based algorithm that alleviate the staircase effect and outperform the state-of-the-art PDE techniques (Weickert, 1998, Ning and Ke, 2012). The proposed PDE models are described in the next section, while the corresponding numerical discretization algorithm is presented in the second. Our image restoration experiments and method comparison are discussed in the fourth section. This article finalizes with conclusions and a list of references.

## 2. A Nonlinear Anisotropic Diffusion Model

We consider a novel nonlinear anisotropic diffusion-based model that provides an efficient noise removal while preserving successfully the image boundaries (Ning and Ke, 2012). Our PDE-based denoising technique is based on the next second-order parabolic equation:

$$\begin{cases} \frac{\partial u}{\partial t} = div\left(\psi_u\left(\|\nabla u\|\right)\nabla u\right) - \nu\left(u - u_0\right) \\ u(0, x, y) = u_0 \end{cases}, (x, y) \in \Omega \tag{1}$$

Where $u_0$ is the degraded image, its domain $\Omega \subset R^2$ and $\nu \in (0,1)$. We construct the next diffusivity (edge-stopping) function $\psi_u : [0, \infty) \to [0, \infty)$, for this restoration scheme:

$$\psi_u(s) = \frac{\lambda}{\left(\frac{s}{K(u)}\right)^2 + K(u)\left|\log_{10}\left(\frac{s}{K(u)}\right)\right|} \tag{2}$$

Where the conductance diffusivity depends on the state of the evolving image at time *t*. We consider a statistics-based automatic computation of this parameter, using the image noise estimation at each time, as follows:

$$K(u) = \xi \cdot \mu\left(\|\nabla u\|\right) + \alpha \cdot ord(u), \tag{3}$$

Where $\xi \in (2,3)$, $\alpha \in (0,1)$, $\mu$ represents the average operator, *ord* (*u*) returns the order of *u* in the evolving sequence.

The proposed diffusivity function $\psi_u$ is properly selected, satisfying the conditions required by an edge-stopping function (Perona and Malik, 1987). So, it is always positive and monotonically decreasing. Since $\frac{\lambda}{\left(\frac{s_1}{K(u)}\right)^2 + K(u)\log_{10}\left(\frac{s_1}{K(u)}\right)} \geq \frac{\lambda}{\left(\frac{s_2}{K(u)}\right)^2 + K(u)\log_{10}\left(\frac{s_2}{K(u)}\right)}, \forall s_1 \leq s_2$, it results that $\psi_u(s_1) \geq \psi_u(s_2), \forall s_1 \leq s_2$. Also, we have $\lim_{s \to \infty} \psi_u(s) = 0$.

Then one can prove the existence and uniqueness of a weak solution in some certain cases. Our PDE model has solution if the function $s \cdot \psi_u(s)$ is monotonically increasing. In order for this to happen, its derivative must be positive: $\psi_u(s) + s\frac{\partial \psi_u(s)}{\partial s} \geq 0$. If $s \leq k(u)$ and $k(u) \geq \ln(10)$ that is a generally verified condition, therefore our PDE model represents a forward parabolic equation that is stable and quite likely to have a solution. A numerical discretization solution for this PDE is described in the following section.

## 3. Numerical Approximation Scheme

We consider a robust numerical discretization of the proposed continuous PDE model. Thus, the discretization of the equation (1) is based on a 4-NN discretization of the Laplacian operator, $\Delta u$ (Weickert, 1998). So, the following numerical approximation is performed on our model:

$$u^{n+1}(x,y)=(1-\nu)u^{n}(x,y)+\varepsilon\sum_{q\in N_p}\psi_u\left(\left|\nabla u_{p,q}(n)\right|\right)\cdot\nabla u_{p,q}(n)+\nu u^{0}(x,y),\ n\in\{0,..,N\} \tag{4}$$

Where $N_p=\{(x-1,y),(x+1,y),(x,y-1),(x,y+1)\}$ represents the set of pixels representing the 4-neighborhood of the pixel described as a pair of coordinates $p$ = [$x$, $y$], $\varepsilon\in(0,1)$, and gradient magnitude in a particular direction at iteration $n$ is computed as:

$$\nabla u_{p,q}(n)=u(q,n)-u(p,n) \tag{5}$$

This iterative restoration algorithm applies the operation (4) on the evolving image for each $n$ from 0 to $N$, where $N$ is the number of iterations providing an optimal denoising. Our restoration scheme is stable and consistent to model given by (1) - (3). It converges fast to the solution of this PDE model, achieving the optimally filtered image $u^{N+1}$ from the noisy image $u^0=u_0$ in a quite low number of iterations, so $N$ takes a rather small value.

## 4. Experiments and Method Comparison

We successfully performed numerous image denoising tests by applying the proposed anisotropic diffusion-based approach. Our restoration scheme was tested on hundreds of images corrupted with various levels of Gaussian noise. It reduced considerably the noise and image blurring, preserved the image details and overcame the staircase effect (Buades et al., 2006).

The following parameters of the PDE model provided the optimal filtering results: $\lambda=1.4,\ \varepsilon=0.3, \xi=2.3, \alpha=0.03,\ N=14.$ The performance of our restoration scheme was assessed by using the PSNR (Peak Signal to Noise Ratio) measure. Method comparison results are registered in Table 1 and displayed in Fig. 1. Our PDE-based filter provides higher PSNR (Peak Signal to Noise Ratio) values (which mean better restoration results) than both Perona-Malik models, TV Denoising and classic [3 x 3] filters, like Average, Gaussian or Median (Jain, 1989).

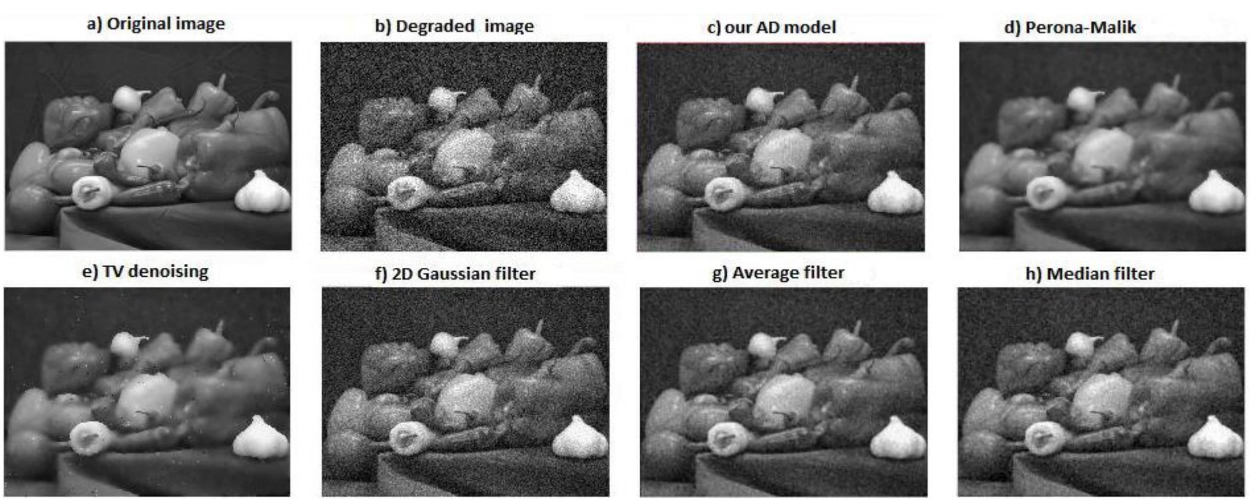


Fig. 1. Denoising results obtained by various schemes.

Table 1. Comparison of the PSNR values for several image restoration methods

| Model | Ours | P-M 1 | P-M 2 | TV | Average | Gaussian | Median |
|---|---|---|---|---|---|---|---|
| PSNR | 27.93(dB) | 26.23(dB) | 26.91(dB) | 27.03(dB) | 26.19(dB) | 22.37(dB) | 24.53(dB) |

The performance of our restoration technique was assessed by using not only this PSNR measure, but also Norm of the Error (NE) Image and Structural Similarity Image Metric (SSIM). All these measures prove that our proposed method provides the best values.

## 5. Conclusions

We have described a PDE-based denoising technique based on nonlinear anisotropic diffusion in this article. This approach provides an efficient feature-preserving image noise removal and overcomes unintended effects, like blurring or staircasing.

The proposed models for the edge-stopping function and its conductance parameter represent the main contributions of this work. The mathematical investigation of the edge-stopping function selection and the well-posedness of this PDE model, as well as the proposed numerical approximation scheme represent also important contributions of this work.

The performed experiments and method comparison results, prove the effectiveness of the developed technique, which outperforms numerous PDE models and conventional image filters. Our future research in this PDE-based image processing domain will focus on developing novel effective restoration models based on higher order PDEs (for example, fourth-order PDE schemes).

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